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A general Halanay inequality and applications

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Abstract: A general Halanay inequality was formulated by a simple method. It could be applied to the study of dynamical behavior of delay different equations. A class of control system was considered with multi-delay and a sufficient condition to guarantee the global exponential stability was established. At last, applications demonstrated part of the effectivity of the development method.

Key words: Halanay inequality; delay different equations; globally exponentially stable

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广义的 Halanay 不等式及其应用

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摘要: 用一种简单的方法证明了广义的 Halanay 不等式，此类不等式可以用来研究时滞微分方程的动态行为. 据此研究了一类多时滞的控制系统并得到系统全局指数稳定的充分条件，并通过应用实例部分验证所得结论的有效性.

关键词: Halanay 不等式; 时滞微分方程; 全局指数稳定

Differential inequalities have played a significant role in the analysis of continuous and discrete time dynamical systems. It is well known that inequalities such as the Halanay inequality[1], is important methods for investigating the dynamical behavior of differential equations. In particular, the Halanay inequality which was first proposed by Halanay[1] has been widely applied to the stability analysis of various delay differential equations, and it has also proved to be a powerful tool in the investigation of distributed delay differential equations[2-5].

Halanay inequality[1] If $v(t) \geqslant 0$, $t \in (-\infty, +\infty)$ and

$$\begin{cases} v'(t) \leqslant av(t) + b[v(t)]_{\tau}, \\ v(t) = \phi(t), \quad t \in [t_0 - \tau, t_0], \end{cases} \tag{1}$$

Where $b \geqslant 0, a < 0, \tau > 0$, $[v(t)]_{\tau} = \sup\limits_{t-\tau(t) \leqslant s \leqslant t} v(s)$. If

$$a + b \leqslant -\sigma < 0, \quad t \geqslant t_0,$$

then there exist constant $\alpha > 0$ and $\beta > 0$ such that

$$v(t) \leqslant \beta e^{-\alpha(t-t_0)}, \quad t \geqslant t_0.$$

1 Main results

The Halanay inequality is very convenient to implement in many real applications. It is worth pointing out that the inequality has been generalized various forms. But it is rare to consider the Halanay inequality with the variable coefficient and multiple time delay. In the following, we consider the Halanay inequality with the variable coefficient and multiple time delay.

Theorem 1 If $v(t) \geqslant 0$, $t \in (-\infty, +\infty)$ and

$$\begin{cases} v'(t) \leqslant \gamma(t) + a(t)v(t) + b(t)[v(t)]_{\tau} + \\ \quad c(t)\int_0^{+\infty} K(s)v(t-s)\,ds, \\ v(t) = \phi(t), \quad t \in (-\infty, t_0], \end{cases} \tag{2}$$

where $\phi(t)$ is bounded and continuous for $t \leqslant t_0$, continuous functions $\gamma(t) \geqslant 0$, $a(t) \leqslant 0$, $b(t) \geqslant 0$, $c(t)$

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$\geqslant 0$, $\gamma^* = \sup_{t_0 \leqslant t < +\infty} \gamma(t)$, the delay kernel $K(\cdot) \geqslant 0$ and $\int_0^{+\infty} K(s)\, e^{\mu s} ds < +\infty$ for some positive number μ.

If there exists $\sigma > 0$ such that

$a(t) + b(t) + c(t) \int_0^{+\infty} K(s)\, ds \leqslant -\sigma < 0, \quad t \geqslant t_0,$

Then there exists a positive number μ^* such that

$$v(t) \leqslant \frac{\gamma^*}{\sigma} + G e^{-\mu^*(t-t_0)}, \quad t \geqslant t_0,$$

Where

$$\mu^* = \inf_{t \geqslant t_0} \left\{ \mu : \mu + a(t) + b(t)\, e^{u\tau(t)} + c(t) \int_0^{+\infty} K(s)\, e^{\mu s} ds = 0 \right\},$$

$$G = \sup_{-\infty < s \leqslant t_0} |\varphi(s)|, \text{and } [v(t)]_\tau = \sup_{t-\tau(t) \leqslant s \leqslant t} v(s), \tau(t) \geqslant 0.$$

Proof We define the function $F(t,\mu)$ by

$$F(t,\mu) = \mu + a(t) + b(t)\, e^{u\tau(t)} + c(t) \int_0^{+\infty} K(s)\, e^{\mu s} ds$$

for any given fixed $t \geqslant t_0$, we can obtain that

$$F(t,0) = a(t) + b(t) + c(t) \int_0^{+\infty} K(s)\, ds \leqslant -\sigma < 0, \lim_{\mu \to +\infty} F(t,\mu) = +\infty,$$

and

$$\frac{\partial F(t,\mu)}{\partial \mu} = 1 + \tau(t)\, b(t)\, e^{u\tau(t)} + c(t) \int_0^{+\infty} s K(s)\, e^{\mu s} ds > 0.$$

Therefore for any given $t \geqslant t_0$, there is a unique positive μ^* such that

$$\mu + a(t) + b(t)\, e^{u\tau(t)} + c(t) \int_0^{+\infty} K(s)\, e^{\mu s} ds = 0.$$

Define

$$u(t) = \begin{cases} \left(v(t) - \dfrac{\gamma^*}{\sigma}\right) e^{\mu^*(t-t_0)}, & t \geqslant t_0, \\ v(t) - \dfrac{\gamma^*}{\sigma}, & -\infty < t < t_0, \end{cases} \tag{3}$$

for $t \geqslant t_0$, we have

$$\frac{du(t)}{dt} = \frac{dv(t)}{dt} e^{\mu^*(t-t_0)} + \left(v(t) - \frac{\gamma^*}{\sigma}\right)\mu^* e^{\mu^*(t-t_0)} \leqslant$$
$$\left[\gamma(t) + a(t)v(t) + b(t)\,[v(t)]_\tau + c(t) \int_0^{+\infty} K(s)\, v(t-s)\, ds\right] e^{\mu^*(t-t_0)} +$$
$$\left(v(t) - \frac{\gamma^*}{\sigma}\right)\mu^* e^{\mu^*(t-t_0)} = \gamma(t)\, e^{\mu^*(t-t_0)} - \frac{\gamma^*}{\sigma}\mu^* e^{\mu^*(t-t_0)} + (a(t) + \mu^*)\frac{\gamma^*}{\sigma} e^{\mu^*(t-t_0)} +$$
$$(a(t) + \mu^*)\, u(t) + b(t)\,[u(t)]_\tau + c(t) \int_0^{+\infty} K(s)\, u(t-s)\, e^{\mu^* s} ds +$$
$$c(t) \int_0^{+\infty} K(s) \frac{\gamma^*}{\sigma} e^{\mu^*(t-t_0)} ds + b(t) \frac{\gamma^*}{\sigma} e^{\mu^*(t-t_0)} =$$
$$\gamma(t)\, e^{\mu^*(t-t_0)} + a(t) \frac{\gamma^*}{\sigma} e^{\mu^*(t-t_0)} + b(t) \frac{\gamma^*}{\sigma} e^{\mu^*(t-t_0)} +$$
$$c(t) \frac{\gamma^*}{\sigma} e^{\mu^*(t-t_0)} \int_0^{+\infty} K(s)\, ds + (a(t) + \mu^*)\, u(t) +$$
$$b(t)\, e^{\mu^* \tau(t)} [u(t)]_\tau + c(t)\, [u(s)]_\infty \int_0^{+\infty} K(s)\, e^{\mu^* s} ds. \tag{4}$$

Since $v(t)$ is continuous and define for $t \in (-\infty, t_0]$, we let

$$\sup_{t_0 - \tau(t_0) \leqslant t \leqslant t_0} \left| v(t) - \frac{\gamma^*}{\sigma} \right| = G.$$

Let $\delta > 1$ be arbitrary, we have $u(t) < \delta G$, for $t \in (-\infty, t_0]$.

We claim

$$u(t) < \delta G, \quad t > t_0.$$

Suppose $u(t) < \delta G$ does not hold for $t > t_0$. Let

$$t_1 = \inf\{t : u(t) = \delta G, t \geqslant t_0\},$$

then

$$\begin{cases} u(t) < \delta G, & -\infty < t \leqslant t_0, \\ u(t_1) = \delta G, \end{cases}$$

and

$$\frac{du(t_1)}{dt} \geqslant 0. \tag{5}$$

We have from (4) and (5)

$$0 \leqslant \frac{du(t_1)}{dt} \leqslant \gamma(t_1)\, e^{\mu^*(t_1-t_0)} +$$
$$\left(a(t_1) + b(t_1) + c(t_1) \int_0^{+\infty} K(s)\, ds\right) \frac{\gamma^*}{\sigma} e^{\mu^*(t_1-t_0)} +$$
$$\left(\mu^* + a(t_1) + b(t_1)\, e^{\mu^* \tau(t_1)} + c(t_1) \int_0^{+\infty} K(s)\, e^{\mu^* s} ds\right) \delta G \leqslant \gamma^* e^{\mu^*(t_1-t_0)} \left(1 + \frac{a(t_1) + b(t_1) + c(t_1) \int_0^{+\infty} K(s)\, ds}{\sigma}\right) < 0$$

This contradicts. So $u(t) < \delta G$, for $t > t_0$.

Let $\delta \to 1$, $u(t) \leqslant G$; so, we have $v(t) \leqslant \frac{\gamma^*}{\sigma} + G e^{-\mu^*(t-t_0)}$.

So, the proof of the Theorem 1 is completed.

Remark 1 If $\gamma(t) \equiv 0$, $c(t) \equiv 0$, then the inequality (2) can be rewritten

$$\begin{cases} v'(t) \leqslant a(t)\, v(t) + b(t)\, [v(t)]_\tau, \\ v(t) = \phi(t), \quad t \in (-\infty, t_0], \end{cases} \tag{6}$$

If $\gamma(t) \equiv 0$, $b(t) \equiv 0$, then the inequality (2) can be rewritten

$$\begin{cases} v'(t) \leqslant a(t)v(t) + c(t)\int_0^{+\infty} K(s)v(t-s)\,ds, \\ v(t) = \phi(t), \quad t \in (-\infty, t_0], \end{cases} \tag{7}$$

If $c(t) \equiv 0$, then the inequality (2) can be rewritten

$$\begin{cases} v'(t) \leqslant \gamma(t) + a(t)v(t) + b(t)\ [v(t)]_\tau, \\ v(t) = \phi(t), \quad t \in (-\infty, t_0], \end{cases} \tag{8}$$

In reference [2-4,6-8], those obtained results exploited the inequality (6), (7) or (8).

Remark 2 The conclusions of these above inequalities are a specialty of our results. In this paper, the proof method of Theorem 1 is different with the reference [6-8]. Our methods is easy to understand and not complicated proof process.

2 Application

Consider the multiple delay differential system

$$\begin{cases} x'(t) = Ax(t) + Bx(t-\tau(t)) + \\ \quad C\int_0^{\sigma(t)} K(s)x(t-s)\,ds, \quad t \geqslant 0, \\ x(t) = \phi(t), \quad -\infty < t \leqslant 0, \end{cases} \tag{9}$$

where $x: R \to R^n$ is the state, $\tau(t) \geqslant 0$, $0 \leqslant \sigma(t) \leqslant \sigma^*$, the delay kernel $K(\cdot) \geqslant 0$ and $\int_0^{+\infty} K(s)e^{\mu s}ds < +\infty$, A, B, C are $n \times n$ matrices.

Theorem 2 The zero solution of (9) is globally exponentially stable if there exist $p < 0$, $q \geqslant 0$, $w \geqslant 0$ and $n \times n$matrices P, Q, $W > 0$ such that

$$Q \leqslant qP,\ W \leqslant wP,\ p + q + w\sigma^* \int_0^{+\infty} K^2(s)\,ds < 0,$$

and
$$\begin{pmatrix} PA + A^TP - pP & PB & PC \\ B^TP & -Q & 0 \\ C^TP & 0 & -W \end{pmatrix} \leqslant 0$$

hold.

Proof Let x be a solution of (9) and define $v(t) = x^T(t)Px(t)$. Then

$$v'(t) = (x'(t))^T Px(t) + x^T(t)Px'(t) = \left(Ax(t) + Bx(t-\tau(t)) + C\int_0^{\sigma(t)} K(s)x(t-s)\,ds\right)^T Px(t) + x^T(t)P\left(Ax(t) + Bx(t-\tau(t)) + C\int_0^{\sigma(t)} K(s)x(t-s)\,ds\right) = (x(t), x(t-\tau(t)), \int_0^{\sigma(t)} K(s)x(t-s)\,ds) \begin{pmatrix} PA + A^TP & PB & PC \\ B^TP & 0 & 0 \\ C^TP & 0 & 0 \end{pmatrix} \times (x(t), x(t-\tau(t)), \int_0^{\sigma(t)} K(s)x(t-s)\,ds)^T =$$

$$(x(t), x(t-\tau(t)), \int_0^{\sigma(t)} K(s)x(t-s)\,ds) \times \begin{pmatrix} PA + A^TP - pP & PB & PC \\ B^TP & -Q & 0 \\ C^TP & 0 & -W \end{pmatrix} \times (x(t), x(t-\tau(t)), \int_0^{\sigma(t)} K(s)x(t-s)\,ds)^T + px^T(t)Px(t) + x^T(t-\tau(t))Qx(t-\tau(t)) + \left(\int_0^{\sigma(t)} K(s)x(t-s)\,ds\right)^T W \int_0^{\sigma(t)} K(s)x(t-s)\,ds \leqslant px^T(t)Px(t) + qx^T(t-\tau(t))Px(t-\tau(t)) + w\sigma^* \int_0^{\sigma(t)} K^2(s)x^T(t-s)Px(t-s)\,ds \leqslant pv(t) + qv(t-\tau) + w\sigma^* \int_0^{\sigma(t)} K^2(s)v(t-s)\,ds,$$

hence the conditions of Theorem 1 are satisfied,

$$v(t) \leqslant Ge^{-\mu^*((t-t_0)}, \quad t \in R^+,$$

Where

$$\mu^* = \inf_{t \geqslant 0}\left\{\mu: \mu + p + qe^{\mu\tau(t)} + w\sigma^* \int_0^{+\infty} K^2(s)e^{\mu s}ds = 0\right\},\ G = \sup_{-\infty < s \leqslant t_0} |\phi(s)|.$$

Therefore, the zero solution of (9) is globally exponentially stable.

Reference:

[1] HALANAY A. Differential equations: stability, oscillations, time lags [M]. New York: Springer-Verlag, 1966.

[2] WANG L S, XU D Y. Global exponential stability of Hopfield reaction diffusion neural networks with time-varying delays [J]. Sci China Ser, 2003, 46 (6) 466-474.

[3] LIU X Z, TEO K L, XU B J. Exponential stability of impulsive high-order Hopfield-type neural networks with time-varying delays [J]. IEEE Transactions on Neural Networks, 2005, 16(6): 1329-1339.

[4] DANIEL W C, LIANG J L, LAM J. Global exponential stability of impulsive high-order BAM neural networks with time-varying delays [J]. Neural Networks, 2006, 19(10): 1581-1590.

[5] XU D Y, YANG Z C. Impulsive delay differential inequality and stability of neural networks [J]. J Math Anal Appl, 2005, 305(1): 107-120.

[6] TIAN H. The exponential asymptotic stability of singularly perturbed delay differential equations with a bounded lag [J]. J Math Anal Appl, 2002, 270(1): 143-149.

[7] HUANG C, HE Y, WANG H. Mean square exponential stability of stochastic recurrent neural networks with time-varying delays [J]. Comput Math Appl, 2008, 56(7): 1773-1778.

[8] WEN L, YU Y, WANG W. Generalized Halanay inequalities for dissipativity of Volterra functional differential equations [J]. J Math Anal Appl, 2008, 347(1): 169-178.

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